JOURNAL OF COMBINATORIAL THEORY, Series A **68**, 33–52 (1994)

P-Partitions and q-Stirling Numbers

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Communicated by George Andrews

Received December 7, 1992

New q-analogs of Stirling numbers of the first and the second kind are derived from a poset on $[2k]$ using Stanley's P-partition theory [*Mem. Amer. Math. Soc.* **119** (1972)]. We also generalize to the poset on the set $[rk]$.

1. INTRODUCTION

The q-analog of Stirling numbers have been studied over the years by Carlitz [1, 2], Gould [7], Milne [9, 10], Sagan [12], and recently by Wachs and White [16] who introduced the p, q-Stirling numbers of the second kind. In [11], I introduced r-multipermutations and studied various statistics on them. Another aspect of the theory of r-multipermutations arises through the theory of P-partitions by Stanley [14]. In this paper we study connections between r-multipermutations and P-partitions, and then q-Stirling numbers. Let us begin with the r-multipermutations.

A multiset is defined as an ordered pair (S, f) where S is a set and f is a function from S to the set of nonnegative integers. If $S = \{m_1, ..., m_r\}$, we write $\{m_1^{f(m_1)}, ..., m_r^{f(m_r)}\}$ for (S, f). We call $f(m_i)$ the multiplicity of m_i. Intuitively, a multiset is a set with possibly repeated elements. Let us begin by considering permutations of multisets (S, f), which are words in which each letter belongs to the set S and for each $m \in S$ the total number of appearances of m in the word is the multiplicity $f(m)$. Thus 1223231 is a permutation of the multiset $\{1^2, 2^3, 3^2\}$. In this paper we consider a special set of permutations of the multiset $[n]^{(r)} = \{1^r, ..., n^r\}$, which is defined as follows.

0097-3165/94 $6.00

DEFINITION 1.1. An r-multipermutation of the set $\{m_1, ..., m_n\}$ is a permutation $a_1 \cdots a_{rn}$ of the multiset $\{m_1^r, ..., m_n^r\}$, where if $i<j<k$ and $a_i = a_k$ then $a_j \geqslant a_i$.

EXAMPLE 1.2. There are three 2-multipermutations 1122, 1221, 2211 of the multiset $[2]^{(2)} = \{1^2, 2^2\}$ and 223332 is a 3-multipermutation of the multiset $\{2^3, 3^3\}$.

Remark. We call r-multipermutations in the case $r=2$ Stirling permutations and we denote the set of ordinary permutation of the set $[n]$ by S_n and the set of r-multipermutations of the multiset $[n]^{(r)}$ by $S_n^{(r)}$.

2. P-PARTITIONS

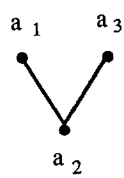
Stanley [14] defines P-partitions as order reversing maps, while I define them as order preserving as Gessel does in [6]. We begin with reviewing his theory briefly. It is a common generalization of the theory of compositions and the theory of partitions, which are defined as follows.

DEFINITION 2.1. A partition of a positive integer n is a solution of $a_1 + a_2 + \cdots + a_k = n$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$, and $a_i \in \mathbf{P}$, where $\mathbf{P}$ is the set of positive integers.

DEFINITION 2.2. A composition of $n \in \mathbf{P}$ is a solution of $a_1 + a_2 + \cdots + a_k = n$, $a_i \in \mathbf{P}$.

EXAMPLE 2.3. There are five partitions of 4: (4), (31), (22), (211), (1111) and there are eight compositions of 4: (4), (13), (31), (22), (112), (121), (211), (1111).

EXAMPLE 2.4. Suppose we have the poset P shown in the following figure and we want to solve $a_1 + a_2 + a_3 = n$, $n \in \mathbf{P}$ with $a_2 \leqslant a_3$ and $a_2 < a_1$, $a_i \in \mathbf{P}$, corresponding to P. Then every solution satisfies either $a_2 < a_1 \leqslant a_3$ or $a_2 \leqslant a_3 < a_1$.



Notice that solutions of these inequalities are disjoint.

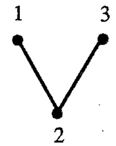
Let us denote the set of natural numbers by $\mathbf{N}$. We now generalize this idea as follows.

DEFINITION 2.5. Let P be a poset on $[k]$ with partial order $\prec$. A P-partition is a function f: $[k] \to \mathbf{N}$ such that

(I) if $i \prec j$ then $f(i) \leqslant f(j)$

(II) if $i \prec j$ and $i > j$ then $f(i) < f(j)$.

EXAMPLE 2.6. Consider the poset P with the following Hasse diagram:



We have $2 \prec 3$ and $2 \prec 1$ and $2 > 1$. Hence a P-partition of the poset is a function f which satisfies $f(2) \leqslant f(3)$ and $f(2) < f(1)$. So if $\sum_{i=1}^{3} f(i) = 4$ and $f(i) > 0$, then there is only one such f and $f(1) = 2$, $f(2) = f(3) = 1$.

DEFINITION 2.7. The separator $\mathscr{S}(P)$ of a poset P is the set of all permutations that extend P to a total order.

EXAMPLE 2.8. In the previous example $\mathscr{S}(P) = \{213, 231\}$.

The following theorem is due to Stanley [14] and Gessel [6].

THEOREM 2.9. *The set of P-partitions is the disjoint union over all* $\pi \in \mathscr{S}(P)$ *of the set of functions* f: $[k] \to \mathbf{N}$ *satisfying* $f(\pi_1) \sim_1 f(\pi_2) \sim_2 \cdots \sim_{k-1} f(\pi_k)$ *where* $\sim_i$ *is* $\leqslant$ *if* $\pi_i < \pi_{i+1}$ *and* $\sim_i$ *is* $<$ *if* $\pi_i > \pi_{i+1}$.

Proof. (By induction on the number of incomparable pairs in P.) If P is a chain, then the theorem is trivial. Suppose i and j are incomparable elements of P. Let P_{ij} be obtained from P by ordering $i \prec j$ and let P_{ji} be obtained from P by ordering $j \prec i$. We want to prove that $\mathscr{S}(P) = \mathscr{S}(P_{ij}) \uplus \mathscr{S}(P_{ji})$, where $\uplus$ denotes disjoint union, and the set of P-partitions is the disjoint union of the set of P_{ij}-partitions and the set of P_{ji}-partitions. First if $\pi \in \mathscr{S}(P)$ then either i comes before j (that is, $\pi \in \mathscr{S}(P_{ij})$) or j comes before i (that is, $\pi \in \mathscr{S}(P_{ji})$). So clearly $\mathscr{S}(P) = \mathscr{S}(P_{ij}) \uplus \mathscr{S}(P_{ji})$. Suppose f is a P-partition and suppose $i < j$. Then either $f(i) \leqslant f(j)$ (i.e., f is a P_{ij}-partition) or $f(j) < f(i)$ (i.e., f is a P_{ji}-partition). So the set of P-partitions is the disjoint union of P_{ij} and P_{ji}. Therefore for any finite P the theorem holds. ∎

DEFINITION 2.10. The order polynomial $\Omega_P(n)$ is the number of P-partitions of a poset P with all parts in $[n]$.

Stanley [14] found a relation between the order polynomial of a poset P and the separator $\mathscr{S}(P)$ which is described in the following theorem. But first we need to introduce the descent set.

DEFINITION 2.11. For a given permutation $\pi = a_1 a_2 \cdots a_n$ we define the descent set of π as $D(\pi) = \{i \in [n-1] \mid a_i > a_{i+1}\}$ and $\operatorname{des}(\pi) = |D(\pi)|$.

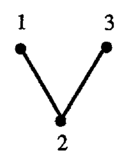
In the following theorem an extra descent at the end of each permutation is counted.

THEOREM 2.12. $\sum_{n=0}^{\infty} \Omega_P(n) t^n = \sum_{\pi \in \mathscr{S}(P)} t^{\operatorname{des}(\pi)}/(1-t)^{k+1}$, $k = |P|$.

Proof. It is enough to consider the case in which P is a chain. Let $\mathscr{S}(P) = \{\pi\}$, where π has d descents including an extra descent at the end. Then $\Omega_P(n)$ is the number of solutions of $1 \leqslant g_1 \leqslant g_2 \leqslant \cdots \leqslant g_k \leqslant n-(d-1)$ which is equal to the number of ways choosing k things from $n-d+1$ with repetitions allowed. So the number is $\binom{n-d+1+k-1}{k} = \binom{n-d+k}{k}$. Therefore

$$\sum_{n=0}^{\infty} \Omega_P(n) t^n = \sum_{n=0}^{\infty} \binom{n-d+k}{k} t^n = \frac{t^d}{(1-t)^{k+1}}. \quad \blacksquare$$

EXAMPLE 2.13. Suppose that we have the following poset P.



Then $\mathscr{S}(P) = \{2 \cdot 13 \cdot, 23 \cdot 1 \cdot\}$ in which a dot represents a descent. So we have

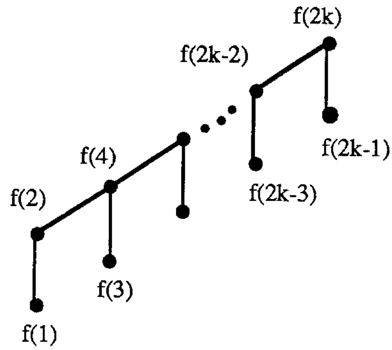
$$\sum_{n=0}^{\infty} \Omega_P(n) t^n = \frac{2t^2}{(1-t)^4} = 2t^2 \sum_{n=0}^{\infty} \binom{n+4-1}{n} t^n = \sum_{n=2}^{\infty} \binom{n+1}{3} t^n.$$

Therefore $\Omega_P(n) = 2\binom{n+1}{3}$.

EXAMPLE 2.14. If P is an antichain

then $\Omega_P(n) = n^m$ and $\mathscr{S}(P)$ is the set of all permutations of the set $[m]$. Then $\sum_{\pi \in \mathscr{S}(P)} t^{\mathrm{des}(\pi)}$ is the ordinary Eulerian polynomial.

Now let us consider the following poset $P_k^{(2)}$ with $2k$ vertices, which I call the Stirling poset.



In a P-partition f, $f(2) \leqslant f(4) \leqslant \cdots \leqslant f(2k-2) \leqslant f(2k)$ and there are $f(2i)$ choices for $f(2i-1)$ $(i = 1, 2, ..., k)$. Therefore order polynomial for the poset P is

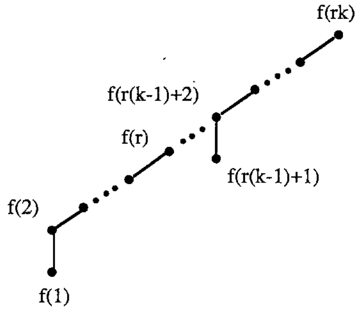
$$\begin{aligned}
\Omega_P(n) &= \sum_{1 \leqslant f(2) \leqslant \cdots \leqslant f(2k) \leqslant n} f(2)\, f(4) \cdots f(2k) \\
&= [x^k] \frac{1}{(1-x)(1-2x)\cdots(1-nx)} \\
&= [x^{k+n}] \frac{x^n}{(1-x)(1-2x)\cdots(1-nx)} \\
&= S(k+n, n).
\end{aligned}$$

Let $\Omega_{P_k^{(2)}}(n)$ and $\Omega_{P_k^{(r)}}(n)$ be order polynomials for Stirling poset $P_k^{(2)}$ and $P_k^{(r)}$ on $[2k]$ and $[rk]$, respectively. Gessel and Stanley [4] obtained the following result for $\Omega_{P_k^{(2)}}(n)$.

PROPOSITION 2.15.

$$\sum_{n=0}^{\infty} \Omega_{P_k^{(2)}}(n)\, t^n = \sum_{\pi \in \mathscr{S}(P_k^{(2)})} t^{\mathrm{des}(\pi)}/(1-t)^{2k+1}.$$

For the poset $P_k^{(r)}$ on $[rk]$ (see the following figure) Schmalzried [13] generalized it as in the next proposition.

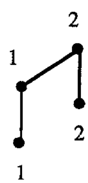


PROPOSITION 2.16.

$$\sum_{n=0}^{\infty} \Omega_{P_k^{(r)}}(n) t^n = \sum_{\pi \in \mathscr{S}(P_k^{(r)})} t^{\operatorname{des}(\pi)}/(1-t)^{rk+1}.$$

For a partial order $\prec$ on the set $[k]$ we consider a label function L from $[k]$ to any set of integers such that 1) if $i \prec j$ then $L(i) \leqslant L(j)$, 2) if $i \prec j$ and $i > j$ then $L(i) > L(j)$, and 3) if $L(i) = L(j)$ then either $i \prec j$ and $i < j$ or $j \prec i$ and $j < i$. Then we can extend L to permutations: if $\pi = (\pi(1), ..., \pi(k))$ then $L(\pi) = (L(\pi(1)), ..., L(\pi(k)))$. Then when L is restricted to $\mathscr{S}(P)$ it is a bijection and moreover descent preserving. Doing this does not change any statistics on the poset. In the Stirling poset on $[2k]$ we label nodes with elements in the multiset $[k]^{(2)}$. Then the separator $\mathscr{S}(P)$ is the set of all permutations of the multiset $[k]^{(2)}$ where the second i comes before the second $i+1$ for $i = 1, ..., k-1$ and $|\mathscr{S}(P)| = 1 \cdot 3 \cdots (2k-1)$.

EXAMPLE 2.17. Let $P_2^{(2)}$ be the following poset with the labeling in which $a_i \prec b_j$ means either $a < b$ and $i \geqslant j$ or $a > b$ and $i \leqslant j$.



Then $\mathscr{S}(P_2^{(2)}) = \{1122, 1212, 2112\}$ in which the second 1 comes before the second 2.

So if we define the Eulerian polynomial $E_{P_k^{(2)}}(t)$ of the poset $P_k^{(2)}$ as $E_{P_k^{(2)}}(t) = \sum_{\pi \in \mathscr{S}(P_k^{(2)})} t^{\operatorname{des}(\pi)}$ then it also counts Stirling permutations of the multiset $[k]^{(2)}$. We expect a bijection between them.

PROPOSITION 2.18. *There is a bijection between the set of Stirling permutations of the multiset* $[k]^{(2)}$ *and* $\mathscr{S}(P_k^{(2)})$ *of the Stirling poset* $P_k^{(2)}$ *that preserves the number of descents.*

Proof. We construct a map ϕ from the set of Stirling permutations of the multiset $[k]^{(2)}$ to the separator of the Stirling poset of $[2k]$. If π is empty then $\phi(\pi)$ is empty. Otherwise π can be obtained from some permutation π' by inserting two k's. Then we construct the bijection ϕ by deriving $\phi(\pi)$ from $\phi(\pi')$ as follows.

1. if k's are inserted in the ith descent of π' to get π then insert one k in the ith descent of $\phi(\pi')$ and the other at the end to get $\phi(\pi)$.

2. if k is inserted in the jth non-descent space of π' to create a descent, then insert one k in the jth non-descent space of $\phi(\pi')$ and the other at the end.

After this process it is obvious that for all i the second $i-1$ comes before the second i and the number of descents is preserved. This bijection is easily reversed. ▮

EXAMPLE 2.19. Suppose that we have $\pi = 34455 \cdot 3 \cdot 122 \cdot 1$.

$$11 \to 122 \cdot 1 \to 33 \cdot 122 \cdot 1 \to 344 \cdot 3 \cdot 122 \cdot 1 \to 34455 \cdot 3 \cdot 122 \cdot 1 = \pi$$

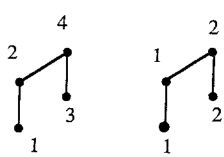
$$11 \to 12 \cdot 12 \to 3 \cdot 12 \cdot 123 \to 3 \cdot 14 \cdot 2 \cdot 1234 \to 35 \cdot 14 \cdot 2 \cdot 12345 = \phi(\pi)$$

We can generalize the above proposition using a similar method. I state it without proof.

THEOREM 2.20. *Let* $P_k^{(r)}$ *be a Stirling poset with rk elements as in the above figure and let* $S_n^{(r)}$ *be the set of r-multipermutations of the multiset* $\{1^r, ..., n^r\}$. *Then there is a bijection between them that preserves the number of descents.*

COROLLARY 2.21. *Also,* $E_{P_k^{(r)}}(t)$ *counts r-multipermutations of the set* $[k]^{(r)}$ *by descents.*

EXAMPLE 2.22. Suppose we have the poset $P_2^{(2)}$ on the set $[4]$ and label its nodes from $\{1^2, 2^2\}$.



Then $\mathscr{S}(P_2^{(2)}) = \{1122\cdot, 2\cdot112\cdot, 12\cdot12\cdot\}$ and $S(n+2,2) = \binom{n+3}{4} + 2\binom{n+2}{4}$. The corresponding Stirling permutations are $\{1122\cdot, 22\cdot11\cdot, 122\cdot1\cdot\}$. So $E_{P_2^{(2)}}(t) = t + 2t^2$, which also counts Stirling permutations of $\{1^2, 2^2\}$ by descents.

3. A q-Analog of the Stirling Numbers of the Second Kind

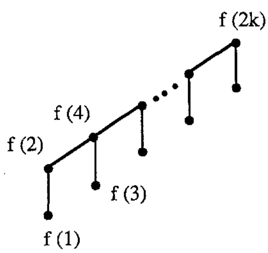
In this section we derive q-Stirling numbers from the Stirling poset $P_k^{(2)}$. Let $T(P, m)$ be the set of all P-partitions with parts in $\{0, 1, ..., m\}$ where P is a poset with p elements. Let

$$U_m(P) = \sum_{\sigma \in T(P,m)} q^{\sigma(1)+\sigma(2)+\cdots+\sigma(p)}.$$

Then $[q^n]\, U_m(P)$ is the number of P-partitions of n with all parts less than or equal to m.

Now for the Stirling poset $P_k^{(2)}$ (see the following figure) we have

$$U_m(P_k^{(2)}) = \sum_{0 \leqslant f(2) \leqslant f(4) \leqslant \cdots \leqslant f(2k) \leqslant m} q^{f(2)+\cdots+f(2k)}(1+q+\cdots+q^{f(2)}) \times (1+q+\cdots+q^{f(4)})\cdots(1+q+\cdots+q^{f(2k)}).$$



So the generating function for $U_m(P_k^{(2)})$ is

$$\sum_{k=0}^{\infty} U_m(P_k^{(2)}) z^k = \frac{1}{1-z} \cdot \frac{1}{1-zq(1+q)} \cdots \frac{1}{1-zq^m(1+q+\cdots+q^m)}. \tag{3.1}$$

We call $U_m(P_k^{(2)})$ a q-Stirling number of the second kind, denoted by $S_q(m+1+k, m+1)$ because if $q=1$ we have

$$\sum_{k=0}^{\infty} U_m(P_k^{(2)}) z^k = \frac{1}{(1-z)(1-2z)\cdots(1-(m+1)z)}$$

$$= \sum_{n > m} S(n, m+1) z^{n-m-1}$$

$$= \sum_{n=0}^{\infty} S(n+m+1, m+1) z^n.$$

Note that $S_q(n, k) = U_{k-1}(P_{n-k}^{(2)})$.

PROPOSITION 3.1.

$$U_m(P_k^{(2)}) = \sum_{i=0}^{k} q^{mi}(1+q+q^2+\cdots+q^m)^i\, U_{m-1}(P_{k-i}^{(2)}).$$

Proof. In a Stirling poset we call minimal elements lower nodes and others upper nodes. For example, nodes labeled $f(2), ..., f(2k)$ are upper nodes in the above figure. Suppose that the top i upper nodes $f(2k), f(2(k-1)), ..., f(2(k-i+1))$ in the above figure are equal to m and the $k-i$ remaining upper nodes are less than or equal to $m-1$. Then $q^{mi}(1+q+\cdots+q^m)^i$ is the generating function for $2i$ nodes (from i upper nodes and the corresponding lower nodes) and the generating function for the remaining poset is $U_{m-1}(P_{k-i}^{(2)})$. ∎

Using the notation $[m]_q = 1+q+\cdots+q^{m-1}$ we express this as follows:

THEOREM 3.2. $S_q(m+1+k, m+1) = S_q(m+k, m) + q^m[m+1]_q\, S_q(m+k, m+1)$.

Proof. Recall

$$\sum_{k=0}^{\infty} U_m(P_k^{(2)}) z^k = \frac{1}{1-z} \cdot \frac{1}{1-zq(1+q)} \cdots \frac{1}{1-zq^m(1+q+\cdots+q^m)}$$

$$= \left(\sum_{k=0}^{\infty} U_{m-1}(P_k^{(2)}) z^k\right) \frac{1}{1-zq^m(1+q+\cdots+q^m)}.$$

Then we have

$$(1-zq^m(1+q+\cdots+q^m)) \sum_{k=0}^{\infty} U_m(P_k^{(2)}) z^k$$

$$= \sum_{k=0}^{\infty} U_{m-1}(P_k^{(2)}) z^k,$$

$$\sum_{k=0}^{\infty} U_m(P_k^{(2)}) z^k - q^m(1+q+\cdots+q^m) \sum_{k=0}^{\infty} U_m(P_k^{(2)}) z^{k+1}$$

$$= \sum_{k=0}^{\infty} U_{m-1}(P_k^{(2)}) z^k.$$

Therefore we obtain

$$U_m(P_k^{(2)}) = U_{m-1}(P_k^{(2)}) + q^m(1+q+\cdots+q^m)\, U_m(P_{k-1}). \quad \blacksquare$$

The theorem shows that for $q=1$ the recurrence relation is reduced to that of the ordinary Stirling numbers of the second kind, that is, $S(m+1+k, m+1) = S(m+k, m) + (m+1)\, S(m+k, m+1)$. Theorem 3.2 can be expressed in a simpler form as

$$S_q(n,k) = S_q(n-1,k-1) + q^{k-1}[k]_q\, S_q(n-1,k) \tag{3.2}$$

Before I give a combinatorial proof of (3.2) let me briefly summarize what has been done with q-Stirling numbers of the second kind.

There are two q-Stirling numbers of the second kind which are defined recursively for nonnegative integers n, k

$$\hat{S}_q(n,k) = \begin{cases} \hat{S}_q(n-1,k-1) + [k]_q\, \hat{S}_q(n-1,k) & \text{if } n,k \geqslant 1 \\ \delta_{n,k} & \text{if } n=0 \text{ or } k=0 \end{cases} \tag{3.3}$$

$$\tilde{S}_q(n,k) = \begin{cases} q^{k-1}\tilde{S}_q(n-1,k-1) + [k]_q\, \tilde{S}_q(n-1,k) & \text{if } n,k \geqslant 1 \\ \delta_{n,k} & \text{if } n=0 \text{ or } k=0, \end{cases} \tag{3.4}$$

where $\delta_{n,k}$ is the Kronecker delta. It is not hard to see that $\tilde{S}_q(n,k) = q^{\binom{k}{2}}\hat{S}_q(n,k)$.

The polynomial (3.3) was introduced by Carlitz [1, 2] in connection with a problem in abelian groups and then by Gould [7]. Milne [9] introduced the polynomial (3.4) in the study of finite dimensional vector space over $GF(q)$ and later in [10] he also showed that it could be viewed combinatorially as a generating function for an inversion statistic on partitions. Sagan [12] proved both equations using a major index for set partitions from the idea that the q-binomial coefficient represent two statistics on permutations, the inversion number and the major index. Wachs and White [16] introduced the p, q-Stirling numbers of the second kind as a generating function for the joint distribution of two "inversion vectors." The p, q-Stirling numbers are defined by

$$S_{p,q}(n,k) = \begin{cases} q^{k-1}S_{p,q}(n-1,k-1) + [k]_{p,q}\, S_{p,q}(n-1,k) & \text{if } n,k \geqslant 1 \\ \delta_{n,k} \quad \text{if } n=0 \text{ or } k=0, \end{cases}$$

where $[k]_{p,q} = p^{k-1} + p^{k-2}q + \cdots + pq^{k-2} + q^{k-1}$.

The relation between this and Eq. (3.2) is that (3.2) is a special case of the p, q-Stirling numbers of the second kind. In fact, it is not hard to show that $S_q(n,k) = q^{-\binom{k}{2}} S_{q,q^2}(n,k)$.

These q-Stirling numbers of the second kind arise as generating functions of certain statistics, i.e. inversions (inv) and major index (maj) on permutations, restricted growth functions, and rook placements. The recursion in Theorem 3.2 can be proved combinatorially using a method involving maj. I follow Sagan's notations in his paper [12].

Let us denote the set of all partitions of the set $[n]$ into k disjoint subsets or blocks by $S([n], k)$. Then the ordinary Stirling numbers of the second kind are $|S([n], k)|$. We will write the blocks of $\pi \in S([n], k)$ as capital letters separated by bars and we always put $\pi = B_1 \mid B_2 \mid \cdots \mid B_k$ in standard form with $\min B_1 < \min B_2 < \cdots < \min B_k$. Let $d_i = |\{b \in B_i \mid b > \min B_{i+1}\}|$. Then the descent multiset DM of π is defined as $DM(\pi) = \{1^{d_1}, ..., (k-1)^{d_{k-1}}\}$ where i^{d_i} means i is repeated d_i times. The major index of π, denoted by $\mathrm{Maj}(\pi)$ (to distinguish it from the usual major index), is the sum of the descents: $\mathrm{Maj}(\pi) = \sum_{i \in DM(\pi)} id_i$. For example, if $\pi = 13 \mid 2 \mid 478 \mid 56$ then $DM(\pi) = \{1^1, 3^2\}$ and $\mathrm{Maj}(\pi) = 7$. Sagan [12] uses this Maj to prove the equation (3.3).

To prove Theorem 3.2 combinatorially, we need to define the bar index, denoted by $\mathrm{bar}(i)$, for $i \in B_j$ $(j = 1, 2, ..., k)$ as follows:

Definition 3.3. If $i = \min B_j$ then $\mathrm{bar}(i) = 0$. If $i \neq \min B_j$ and if all $r < i$ are in $B_1 \cup \cdots \cup B_l$ $(l \leqslant k)$, where l is the smallest with this property, then $\mathrm{bar}(i) = l - 1$. For $\pi \in S([n], k)$, $\mathrm{bar}(\pi) = \sum_{i=1}^{n} \mathrm{bar}(i)$.

Example 3.4. Let us look at the following two examples.

(a) If $\pi = 1 \mid 24 \mid 3$ then $\mathrm{bar}(1) = \mathrm{bar}(2) = \mathrm{bar}(3) = 0$ and $\mathrm{bar}(4) = 3 - 1 = 2$.

(b) If $\pi = 13 \mid 2 \mid 478 \mid 56$ then $\mathrm{bar}(\pi) = \mathrm{bar}(3) + \mathrm{bar}(6) + \mathrm{bar}(7) + \mathrm{bar}(8) = 1 + 3 + 3 + 3 = 10$.

Theorem 3.5. *If $n \geqslant 1$ and $k \geqslant 1$, then*

$$S_q(n, k) = \sum_{\pi \in S([n], k)} q^{\mathrm{Maj}(\pi) + \mathrm{bar}(\pi)} \tag{3.5}$$

Proof. The right-hand side obviously satisfies the boundary conditions that $S_q(n, k)$ does. Let $\pi \in S([n], k)$ and let π' be the partition obtained by deleting n from π. If $\{n\}$ is a singleton block of $\pi = B_1 \mid \cdots \mid B_k$ then B_k should be $\{n\}$ because π is in standard form. Thus $\pi' \in S([n-1], k-1)$ so $\mathrm{Maj}(\pi) = \mathrm{Maj}(\pi')$ and $\mathrm{bar}(\pi) = \mathrm{bar}(\pi')$. If n is contained in B_i for some i, then

$$\mathrm{Maj}(\pi) = \begin{cases} \mathrm{Maj}(\pi') + i & \text{if } \ 1 \leqslant i < k \\ \mathrm{Maj}(\pi') & \text{if } \ i = k \end{cases}$$

and $\mathrm{bar}(\pi) = \mathrm{bar}(\pi') + (k-1)$. Thus we have

$$\sum_{\pi \in S([n], k)} q^{\mathrm{Maj}(\pi) + \mathrm{bar}(\pi)}$$

$$= \sum_{\pi' \in S([n-1], k-1)} q^{\mathrm{Maj}(\pi') + \mathrm{bar}(\pi')}$$

$$+ \sum_{i=0}^{k-1} \sum_{\pi' \in S([n-1], k)} q^{\mathrm{Maj}(\pi') + i + \mathrm{bar}(\pi') + (k-1)}$$

$$= \sum_{\pi' \in S([n-1], k-1)} q^{\mathrm{Maj}(\pi') + \mathrm{bar}(\pi')}$$

$$+ \sum_{\pi' \in S([n-1], k)} q^{\mathrm{Maj}(\pi') + \mathrm{bar}(\pi')} \sum_{i=0}^{k-1} q^{i + (k-1)}$$

$$= \sum_{\pi' \in S([n-1], k-1)} q^{\mathrm{Maj}(\pi') + \mathrm{bar}(\pi')} + q^{k-1}[k]_q \sum_{\pi' \in S([n-1], k)} q^{\mathrm{Maj}(\pi') + \mathrm{bar}(\pi')}$$

This shows that the right-hand side of (3.5) satisfies the recursion of $S_q(n, k)$. ∎

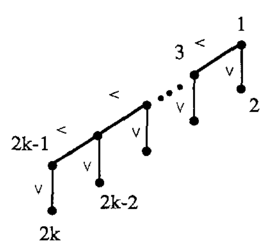
To keep track of $\mathrm{bar}(\pi)$ we weight it by p. Then it is not hard to see that:

PROPOSITION 3.6. *Let* $\underline{S}_{p,q}(n, k) = \sum_{\pi \in S([n], k)} q^{\mathrm{Maj}(\pi)} p^{\mathrm{bar}(\pi)}$. *Then*

$$\underline{S}_{p,q}(n, k) = (pq)^{-\binom{k}{2}} S_{pq,p}(n, k).$$

4. A q-ANALOG OF THE STIRLING NUMBERS OF THE FIRST KIND

Let $\bar{P}_k^{(2)}$ be the strictly labeled Stirling poset illustrated as in the following Hasse diagram.



Then

$$U_m(\bar{P}_k) = \sum_{0 \leqslant f(1) \leqslant f(3) \leqslant \cdots \leqslant f(2k-1) \leqslant m} q^{f(1) + \cdots + f(2k-1)} \times (1 + q + \cdots + q^{f(1)-1}) \times (1 + q + \cdots + q^{f(2)-1}) \cdots (1 + q + \cdots + q^{f(2k-1)-1}).$$

So the generating function for $U_m(\bar{P}_k)$ is

$$\sum_{k=0}^{\infty} U_m(\bar{P}_k) z^k = (1 + zq)(1 + zq^2(1+q)) \cdots (1 + zq^m(1 + q + \cdots + q^{m-1})) \tag{4.6}$$

We call $U_m(\bar{P}_k)$ a q-analog of the signless Stirling number of the first kind, denoted by $c_q(m+1, m+1-k)$ since if $q = 1$ we have

$$\sum_{k=0}^{\infty} U_m(\bar{P}_k) z^k = (1+z)(1+2z) \cdots (1+mz) = \sum_{k=0}^{m} c(m+1, m+1-k) z^k,$$

where $c(m+1, m+1-k)$ is the signless Stirling numbers of the first kind. Note that $c_q(n, k) = U_{n-1}(\bar{P}_{n-k})$.

As in the previous section, we easily obtain the recurrence relation for c_q from

$$\begin{aligned} \sum_{k=0}^{\infty} U_m(\bar{P}_k) z^k &= \left(\sum_{k=0}^{\infty} U_{m-1}(\bar{P}_k) z^k \right) (1 + zq^m[m]_q) \\ &= \sum_{k=0}^{\infty} U_{m-1}(\bar{P}_k) z^k + q^m[m]_q \sum_{k=0}^{\infty} U_{m-1}(\bar{P}_k) z^{k+1}. \end{aligned}$$

Thus we get

$$U_m(\bar{P}_k) = U_{m-1}(\bar{P}_k) + q^m[m]_q\, U_{m-1}(\bar{P}_{k-1}),$$

which leads us to the following recurrence relation.

THEOREM 4.1. $c_q(m+1, m+1-k) = {}_q(m, m-k) + q^m[m]_q\, c_q(m, m-k+1)$.

It can be written simply as

$$c_q(n, k) = c_q(n-1, k-1) + q^{n-1}[n-1]_q\, c_q(n-1, k). \tag{4.7}$$

If $q = 1$ we have $c(n, k) = c(n-1, k-1) + (n-1)\, c(n-1, k)$, $n, k \geqslant 1$ and $c(0, 0) = 1$ which is the recurrence relation of the ordinary signless Stirling numbers of the first kind.

Gould [7] defines q-Stirling numbers of the first kind. These have been given combinatorial interpretations in terms of rook placements by Garsia and Remmel [3] and in terms of permutations by Gessel [5]. De Médicis and Leroux [8] give a combinatorial interpretation to the p, q-Stirling numbers using "0-1 tableaux." Their recurrence relation is

$$c_{p,q}(n, k) = c_{p,q}(n-1, k-1) + [n-1]_{p,q}\, c_{p,q}(n-1, k).$$

If $p = 1$ then it is the same as Gould's. It is easy to see that the relation with the recursion (4.7) is $c_q(n, k) = q^{n-k} c_{q,q^2}(n, k)$.

The generating function for c_q is the following, which leads us to a q-analog of Stirling numbers of the first kind, s_q. In the generating function

$$\sum_{k=0}^{\infty} c_q(m+1, m+1-k) z^k = (1+zq)(1+zq^2(1+q)) \cdots (1+zq^m[m]_q)$$

we replace z by $-z$. Then we have

$$\sum_{k=0}^{\infty} c_q(m+1, m+1-k)(-1)^k z^k = (1-zq)(1-zq^2(1+q)) \cdots (1-zq^m[m]_q).$$

Thus

$$c_q(m+1, m+1-k)(-1)^k = c_q(m, m-k)(-1)^k - q^m[m]_q\, c_q(m, m-k+1)(-1)^k.$$

Now let $c_q(n, k)(-1)^{n-k} = s_q(n, k)$. Then the above equation becomes:

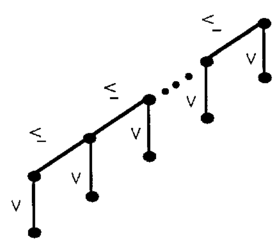
THEOREM 4.2.

$$s_q(n, k) = s_q(n-1, k-1) - q^{n-1}[n-1]_q\, s_q(n-1, k). \tag{4.8}$$

LEMMA 4.3. *Using the method we used to derive $U_m(P_k^{(2)})$ and $U_m(\bar{P}_k)$ we get the following*:

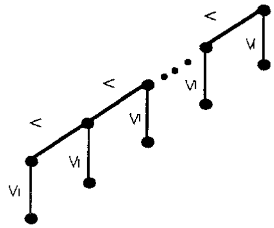
(1) *The generating function analogous to the equation* 3.1 *for the following poset is*

$$\frac{1}{(1-zq)(1-zq^2(1+q)) \cdots (1-zq^m(1+q+\cdots+q^{m-1}))}.$$



(2) *The analogous generating function of the following poset is*

$$(1+zq(1+q))(1+zq^2(1+q+q^2))\cdots(1+zq^m(1+q+\cdots+q^m)).$$



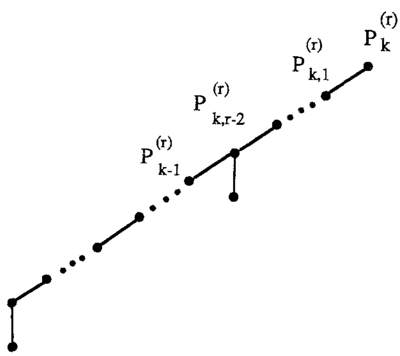
5. THE STIRLING POSET ON $[rk]$

Let $P_k^{(r)}$ be the Stirling poset on $[rk]$ as in the following figure. Let $U_m(P_k^{(r)})=\sum_{\pi\in T(P_k^{(r)},m)} q^{\pi(1)+\cdots+\pi(rk)}$. Then we want to find out the generating function for $U_m(P_k^{(r)})$.

Let us consider $U_{m+1}(P_k^{(r)})-U_m(P_k^{(r)})$. This is the difference of two generating functions for the same poset $P_k^{(r)}$. But one has the condition that all parts are at most $m+1$ and the other has all parts at most m. Let $P_{k,1}^{(r)}$ denote the poset $P_k^{(r)}$ with the top upper node removed. Then its generating function with all parts at most $m+1$ is $U_{m+1}(P_{k,1}^{(r)})$. The weight of the removed node is q^{m+1}. Therefore we have the following recurrence:

$$U_{m+1}(P_k^{(r)})-U_m(P_k^{(r)})=q^{m+1}U_{m+1}(P_{k,1}^{(r)}).$$

From this observation we obtain a recursion for $U_m(P_k^{(r)})$.



DEFINITION 5.1. The shift operator E is defined on the functions f on $\mathbf{N}$ as $(Ef)(m)=f(m+1)$ for $m\in\mathbf{N}$. So $E(U_m(P_k^{(r)}))=U_{m+1}(P_k^{(r)})$. The identity operator is denoted by I. Finally, the q-difference operator Δ_q^i is defined as $\Delta_q^i=\prod_{k=0}^{i-2}(E-q^iI)$ for $i\geqslant 2$.

THEOREM 5.2.

$$\Delta_q^{(r)} U_m(P_k^{(r)}) = q^{(r-1)(m+r-1)}(1+q+\cdots+q^{m+r-1})\, U_{m+r-1}(P_{k-1}^{(r)}), \quad r \geqslant 2.$$

Proof. Let $P_{k,i}^{(r)}$ be the poset obtained from $P_k^{(r)}$ by removing i upper nodes from the top of its Hasse diagram ($i=1, 2, ..., r-2$). Then we have the following recurrences:

$$\begin{aligned}
U_{m+1}(P_k^{(r)}) - U_m(P_k^{(r)}) &= q^{m+1} U_{m+1}(P_{k,1}^{(r)}) \\
&\cdots \\
U_{m+1}(P_{k,i}^{(r)}) - U_m(P_{k,i}^{(r)}) &= q^{m+1} U_{m+1}(P_{k,i+1}^{(r)}) \\
&\cdots \\
U_{m+1}(P_{k,r-2}^{(r)}) - U_m(P_{k,r-2}^{(r)}) &= q^{m+1}(1+q+\cdots+q^{m+1})\, U_{m+1}(P_{k-1}^{(r)})
\end{aligned} \tag{5.9}$$

Now we apply Δ_q. My claim is that for $i=2, ..., r$

$$\Delta_q^{(i)} U_m(P_k^{(r)}) = \prod_{k=0}^{i-2} (E - q^k I)\, U_m(P_k^{(r)}) = q^{(i-1)(m+i-1)} U_{m+i-1}(P_{k,i-1}^{(r)}) \tag{5.10}$$

For $i=2$ it is obviously true. Suppose that the equation (5.10) is true for up to i.

$$\begin{aligned}
&(E - q^{i-1} I) \prod_{k=0}^{i-2} (E - q^k I)\, U_m(P_k^{(r)}) \\
&= (E - q^{i-1} I)\, q^{(i-1)(m+i-1)} U_{m+i-1}(P_{k,i-1}^{(r)}) \\
&= q^{(i-1)(m+i-1)} q^{i-1} [U_{m+i}(P_{k,i}^{(r)}) - U_{m+i}(P_{k,i}^{(r)})] \\
&= q^{(i-1)(m+i)} [q^{m+i} U_{m+i}(P_{k,i+1}^{(r)})] \\
&= q^{i(m+i)} U_{m+i}(P_{k,i+1}^{(r)})
\end{aligned}$$

Therefore

$$\Delta_q^{(i+1)} U_m(P_k^{(r)}) = (E - q^{i-1} I) \prod_{k=0}^{i-2} (E - q^k I)\, U_m(P_k^{(r)}) = q^{i(m+i)} U_{m+i}(P_{k,i+1}^{(r)})$$

So our claim is true for all $i=2, ..., r$.

Now we have

$$\Delta_q^{(r-1)} = \prod_{k=0}^{r-3} (E - q^k I)\, U_{m+1}(P_k^{(r)}) = q^{(r-2)(m+r-2)} U_{m+r-2}(P_{k,r-2}^{(r)}).$$

From the Eq. (5.9),

$$\begin{aligned}
(E-q^{r-2}I)\, q^{(r-2)(m+r-2)}U_{m+r-2}(P^{(r)}_{k,r-2})
&= q^{(r-2)(m+r-2)+(r-2)}[U_{m+r-1}(P^{(r)}_{k,r-2}) - U_{m+r-2}(P^{(r)}_{k,r-2})] \\
&= q^{(r-2)(m+r-1)}[q^{m+r-1}(1+q+\cdots+q^{m+r-1})\, U_{m+r-1}(P^{(r)}_{k-1})].
\end{aligned}$$

So the proof follows. ∎

Remark. The case $r=2$ of this formula is exactly the same as Theorem 3.2 (m is shifted by 1). Note that the $q=1$ case is in Gessel and Stanley's paper [4].

Also we can generalize the identity (4.6) as follows. Let $\bar{P}^{(r)}_k$ be the Stirling poset of $[rk]$ with strictly increasing order. Let $\bar{P}^{(r)}_{k,i}$ be the poset obtained by removing i upper nodes from the top of the Hasse diagram of $\bar{P}^{(r)}_k$. Then we have the following recurrences:

$$\begin{aligned}
U_m(\bar{P}^{(r)}_k) - U_{m-1}(\bar{P}^{(r)}_k) &= q^m U_{m-1}(\bar{P}^{(r)}_{k,1}) \\
&\cdots \\
U_{m-i}(\bar{P}^{(r)}_{k,i}) - U_{m-i-1}(\bar{P}^{(r)}_{k,i}) &= q^{m-i} U_{m-i-1}(\bar{P}^{(r)}_{k,i+1}) \\
&\cdots \\
U_{m-r+2}(\bar{P}^{(r)}_{k,r-2}) - U_{m-r+1}(\bar{P}^{(r)}_{k,r-2}) &= q^{m-r+2}(1+q+\cdots+q^{m-r+1}) \\
&\quad \times U_{m-r+1}(\bar{P}^{(r)}_{k-1}).
\end{aligned}$$

Now we define some other operators:

DEFINITION 5.3. The down operator Λ is defined on the set of functions f on $\mathbf{N}$ as $(\Lambda f)(m) = f(m-1)$. Then $\Lambda U_m(\bar{P}^{(r)}_k) = U_{m-1}(\bar{P}^{(r)}_k)$. The identity operator is denoted by I. The q-down difference operator ∇_q is defined by $\nabla^{(i)}_q = \prod_{k=0}^{-i+2} (q^{-k}I - \Lambda)$ for $i \geqslant 2$.

THEOREM 5.4.

$$\nabla^{(r)}_q U_m(\bar{P}^{(r)}_k) = q^{(r-1)(m-r+2)}(1+q+\cdots+q^{m-r+1})\, U_{m-r+1}(\bar{P}^{(r)}_{k-1}).$$

Proof. The proof is similar to that of Theorem 5.2. For $i=2, ..., r$ one can easily show that

$$\nabla^{(i)}_q U_m(\bar{P}^{(r)}_k) = q^{(i-1)(m-i+2)} U_{m-i+1}(\bar{P}^{(r)}_{k,i-1}). \tag{5.11}$$

Thus

$$\begin{aligned}
&\nabla_q^{(r-1)} U_m(\bar{P}_k^{(r)})\\
&\quad = q^{(r-2)(m-r+3)} U_{m-r+2}(\bar{P}_{k,r-2}^{(r)})\\
&\quad = (q^{-r+2}I - \Delta)\, q^{(r-2)(m-r+3)} U_{m-r+2}(\bar{P}_{k,r-2}^{(r)})\\
&\quad = q^{(r-2)(m-r+3)+(-r+2)} [U_{m-r+2}(\bar{P}_{k,r-2}^{(r)})\, U_{m-r+1}(\bar{P}_{k,r-2}^{(r)})]\\
&\quad = q^{(r-2)(m-r+2)} [q^{m-r+2}(1+q+\cdots+q^{m-r+1} U_{m-r+1}(\bar{P}_{k-1}^{(r)})]. \quad \blacksquare
\end{aligned}$$

Remark. For the case of $r=2$ the formula is identical to Theorem 4 in the previous section.

6. Comajor Index

Using the theorem by Stanley [15, 4.5.4 Theorem, p. 214] one obtains the generating function for counting permutations by major index. His P-partitions are order-reversing unlike here so an order-preserving version of his theorem is needed. Then we get the generating function for counting permutations by comajor index which is defined as follows.

Definition 6.1. Let π be a permutation of p letters. Then the comajor index of π denoted by $\operatorname{comaj}(\pi)$ is defined by $\operatorname{comaj}(\pi) = \sum_{i \in D(\pi)} (p-i)$.

Definition 6.2. Let $G_P(q_1, ..., q_k) = \sum_{\sigma \in T(P)} q_1^{\sigma(1)} \cdots q_k^{\sigma(k)}$, where P is a poset with k elements and $T(P)$ is the set of all P-partitions.

Then we have the order preserving version of Stanley's theorem [15, 4.5.4 Theorem, p. 214].

Lemma 6.3.

$$G_P(q_1, ..., q_k) = \sum_{\pi \in \mathscr{S}(P)} \frac{\prod_{j \in D(\pi)} q_{\pi(k)}\, q_{\pi(k-1)} \cdots q_{\pi(k-j+1)}}{\prod_{i=1}^{k} (1 - q_{\pi(k)}\, q_{\pi(k-1)} \cdots q_{\pi(k-i+1)})}.$$

Hence if we set all $q_i = q$, then

$$\begin{aligned}
G_P(q) &= \sum_{\sigma \in T(P)} q^{\sigma(1)+\cdots+\sigma(k)}\\
&= \sum_{\pi \in \mathscr{S}(P)} \frac{\prod_{j \in D(\pi)} q^{k-j}}{\prod_{i=1}^{k} (1-q^i)}\\
&= \frac{\sum_{\pi \in \mathscr{S}(P)} q^{\operatorname{comaj}(\pi)}}{\prod_{i=1}^{k} (1-q^i)}.
\end{aligned}$$

From Lemma 6.3 we have

$$\sum_{\sigma \in T(P)} q^{\sigma(1)+\cdots+\sigma(2k)} = \frac{\sum_{\pi \in \mathscr{S}(P_k^{(2)})} q^{\mathrm{comaj}(\pi)}}{\prod_{i=1}^{2k} (1-q^i)}.$$

Notice that left-hand side is just $U_m(P_k^{(2)})$. Since $\sum_{k=0}^{\infty} U_m(P_k^{(2)}) z^k = \prod_{i=0}^{m} (1 - zq^i(1+q+\cdots+q^i)^{-1}$, we obtain

PROPOSITION 6.4.

$$\sum_{\pi \in \mathscr{S}(P_k^{(2)})} q^{\mathrm{comaj}(\pi)} = \prod_{i=1}^{2k} (1-q^i)[z^k] \prod_{i=0}^{\infty} (1 - zq^i(1+q+\cdots+q^i))^{-1}. \quad \blacksquare$$

APPENDIX

Symbol	Meaning
$[n]^{(r)}$	The multiset with r copies of each $i \in [n] = \{1, ..., n\}$
$S_n^{(r)}$	The set of r-multipermutations of $[n]^{(r)}$
$\mathbf{P}$	The set of all positive integers
$\mathbf{N}$	The set of all natural numbers
$\mathscr{S}(P)$	The separator of a poset P
$\Omega_P(n)$	The order polynomial
$D(\pi)$	The set $\{i \in [n-1] \mid a_i > a_{i+1}\}$, where $\pi = a_1 a_2 \cdots a_n$
$\mathrm{des}(\pi)$	$\lvert D(\pi) \rvert$
$\Omega_{P_k^{(r)}}(n)$	The order polynomials for Stirling poset $P_k^{(r)}$ on $[rk]$
$S(n, k)$	The Stirling numbers of the second kind
$E_{P_k^{(r)}}(t)$	The Eulerian polynomial for Stirling poset $P_k^{(r)}$ of the set $[k]^{(r)}$
$T(P, m)$	The set of all P-partitions with parts in $\{0, 1, ..., m\}$
$U_m(P)$	The sum $\sum_{\sigma \in T(P,m)} q^{\sigma(1)+\sigma(2)+\cdots+\sigma(p)}$
$S_q(n, k)$	$= U_{k-1}(P_{n-k}^{(2)})$
$[m]_q$	$= 1 + q + \cdots + q^{m-1}$
$\bar{P}_k^{(2)}$	The strictly labeled Stirling poset
$c(n, k)$	The signless Stirling numbers of the first kind
$c_q(n, k)$	$= U_{n-1}(\bar{P}_{n-k})$
E	The shift operator
Δ_q^i	The q-difference operator
$\bar{P}_k^{(r)}$	The Stirling poset of $[rk]$ with strictly increasing order
Δ	The down operator
∇_q	The q-down difference operator
$\mathrm{comaj}(\pi)$	The comajor index of π

ACKNOWLEDGMENTS

I express my deepest gratitude to Professor Ira Gessel for his guidance and encouragement throughout this work.

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